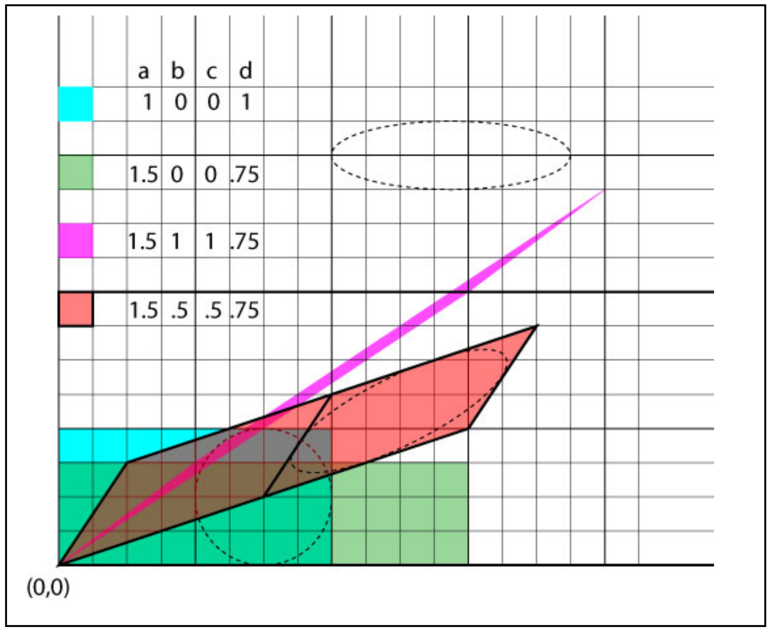
Consider:

$x_1 = ax + by \quad y_1 = cx + dy \quad a \neq 0,\ d \neq 0$

Examples on the right show this for an original body in blue, and three strained bodies in green and purple, and red.



This represents the General Linear Transformation. We first wish to make sure that a deformation of this type gives Homogeneous Strain, where straight, parallel lines remain straight and parallel. We note that we can solve the equations above to get x and y in terms of x_1 and y_1:

$$\begin{array}{cc}
cx_1 = cax + cby & dx_1 = dax + dby \\
\underline{-(ay_1 = acx + ady)} & \underline{-(by_1 = bcx + bdy)} \\
cx_1 - ay_1 = (cb - ad)y & dx_1 - by_1 = (ad - bc)x \\
\text{so} & \text{so} \\
y = \dfrac{cx_1 - ay_1}{cb - ad} & x = \dfrac{dx_1 - by_1}{ad - bc}
\end{array}$$

Now, suppose we begin with a straight line in the undeformed space having an equation y = mx + k. According to our transformation, this line will have an equation in the deformed state of

$$\frac{cx_1 - ay_1}{cb - ad} = \frac{mdx_1 - mby_1}{ad - bc} + k$$

multiply the left hand side by $\dfrac{-1}{-1}$

$$\frac{ay_1 - cx_1}{ad - bc} = \frac{mdx_1 - mby_1}{ad - bc} + k$$

Multiply both sides by the denominator:

$$ay_1 - cx_1 = mdx_1 - mby_1 + k(ad - bc)$$

$$y_1 = \left(\frac{c + md}{a + mb}\right)x_1 + k\frac{ad - bc}{a + mb}$$

This is the equation for a straight line, so this transformation results in a straight line, and all lines with an undeformed slope of “m” will have a slope in the deformed state of $\left(\dfrac{c + md}{a + mb}\right)$ and so will be parallel.

How will the unit circle, $x^2 + y^2 = 1$, transform by this general linear transformation? Substituting our results for x and y we find:

$$c^2 x_1^2 - 2cax_1y_1 + a^2y_1^2 + d^2x_1^2 - 2bdx_1y_1 + b^2y_1^2 = a^2d^2 - 2abcd + b^2c^2$$

or

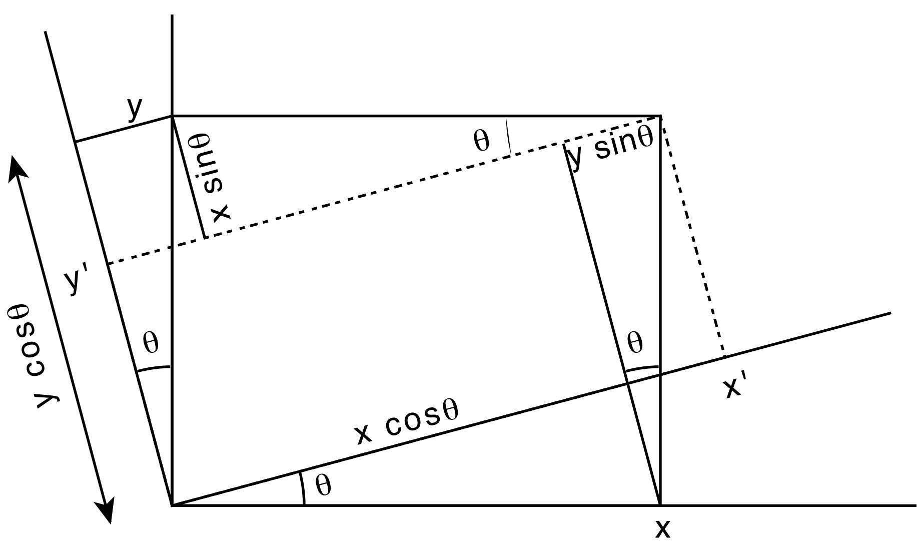
$$(c^2 + d^2)x_1^2 - 2(ac+bd)x_1y_1 + (a^2 + b^2)y_1^2 = (ad - bc)^2$$

This equation looks messy, and we may "simplify" it by defining some new constants in terms of our old constants... Let

$$\frac{c^2 + d^2}{(ad - bc)^2} = \mathbf{l} \qquad \frac{ac + bd}{(ad - bc)^2} = \mathbf{m} \qquad \frac{a^2 + b^2}{(ad - bc)^2} = \mathbf{n}$$

Now our equation looks like: $\boldsymbol{l}\, x_1^2 - 2\boldsymbol{m}\, xy + \boldsymbol{n}\, y_1^2 = 1$

This may be "simplified" further, if we introduce new axes which have undergone rotation:



$$x' = x \cos\theta + y \sin\theta$$

$$y' = y \cos\theta - x \sin\theta$$

Then:

$$(x')^2 = x^2\cos^2\theta + 2xy \sin\theta \cos\theta + y^2\sin^2\theta$$

$$(y')^2 = y^2\cos^2\theta - 2xy \sin\theta \cos\theta + x^2\sin^2\theta$$

$$(x'y') = xy \cos^2\theta + y^2 \sin\theta \cos\theta - x^2 \sin\theta \cos\theta - xy \sin^2\theta$$

big deal....

If we look at our equation for the deformed circle, and assume that it is in the rotated coordinated system, we can find what rotation, θ, is needed to get rid of the coefficient of the xy term. So our equation becomes

$$l(x')^2 - 2m(x'y') + n(y')^2 = 1$$

or $lx^2\cos^2\theta$ + $\mathbf{2lxy\sin\theta\cos\theta}$ + $ly^2\sin^2\theta$ - $\mathbf{2mxy\cos^2\theta}$ - $2my^2\sin\theta\cos\theta$ + $2mx^2\sin\theta\cos\theta$ + $\mathbf{2mxy\sin^2\theta}$ + $ny^2\cos^2\theta$ -$\mathbf{2nxy\sin\theta\cos\theta}$ + $nx^2\sin^2\theta$ = 1

Collecting the terms in bold (those with xy in them), we can set it equal to zero so that it will vanish for all x and y:

$$2lxy\sin\theta\cos\theta - 2mxy\cos^2\theta + 2mxy\sin^2\theta - 2nxy\sin\theta\cos\theta = 0$$

Divide through by 2xy to get $l\sin\theta\cos\theta - m\cos^2\theta + m\sin^2\theta - n\sin\theta\cos\theta = 0$

Or

$$(l - n)\sin\theta\cos\theta - m(\cos^2\theta - \sin^2\theta) = 0$$

$$\frac{l-n}{m} = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos 2\theta}{\frac{\sin 2\theta}{2}} = \frac{2}{\tan 2\theta}$$

Going back to where we defined l, m and n, we see that for the coefficient of the xy term to vanish:

$$\frac{c^2 + d^2 - a^2 - b^2}{ac + bd} = \frac{2}{\tan 2\theta}$$

$$\tan 2\theta = \frac{2\cdot(ac+bd)}{c^2 + d^2 - a^2 - b^2}$$

Finally:

$$\theta = \frac{1}{2}\arctan\frac{2\cdot(ac+bd)}{c^2 + d^2 - a^2 - b^2}$$

For our purple body,

θ = 0.5 arctan { 2 * (1.5 + .75)} / (1 + 0.5625 - 2.25 - 1) = -34.722°

arctan(11/16) = 34.51° so this looks good

For our red body,

θ = 0.5 arctan { 2 * (.75 + .375)} / (.25 + 0.5625 - 2.25 - .25) = -26.565°

$$\frac{c^2 + d^2}{(ad - bc)^2} = \mathbf{l} \qquad \frac{ac + bd}{(ad - bc)^2} = \mathbf{m} \qquad \frac{a^2 + b^2}{(ad - bc)^2} = \mathbf{n}$$

denominator = $(1.125 - .25)^2 = .765625$

l = 1.0612 m = 1.46938 n = 3.2653 $\sin^2\theta = 0.2$ $\cos^2\theta = 0.8$ $\sin\theta\cos\theta = -0.4$

We had: $lx^2\cos^2\theta + ly^2\sin^2\theta - 2my^2\sin\theta\cos\theta + 2mx^2\sin\theta\cos\theta + ny^2\cos^2\theta + nx^2\sin^2\theta = 1$

which becomes

$$(.8 * 1.0612 + 2 *(- .4) * 1.469 + .2 * 3.265)x^2 + (.2 * 1.0612 - 2 * (- .4)* 1.469 + .8 * 3.265)y^2 = 1$$

$$0.3265\ x^2 + 4\ y^2 = 1$$

$$(x/1.75)^2 + (y/.5)^2 = 1$$

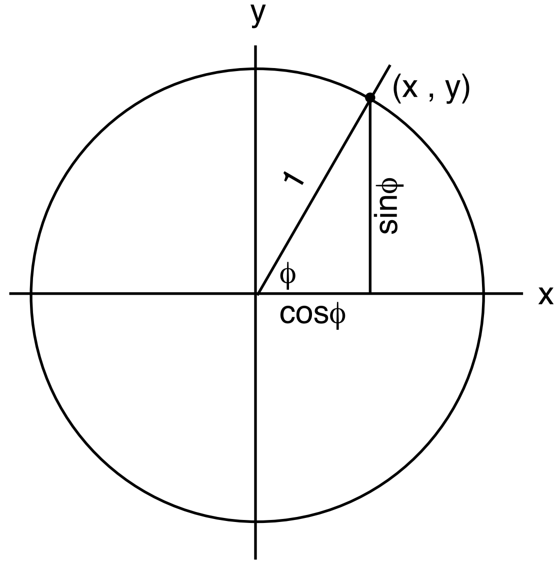
These results are for a unit circle. The circle drawn on the original blue rectangle on the first page had a diameter of 4, however.

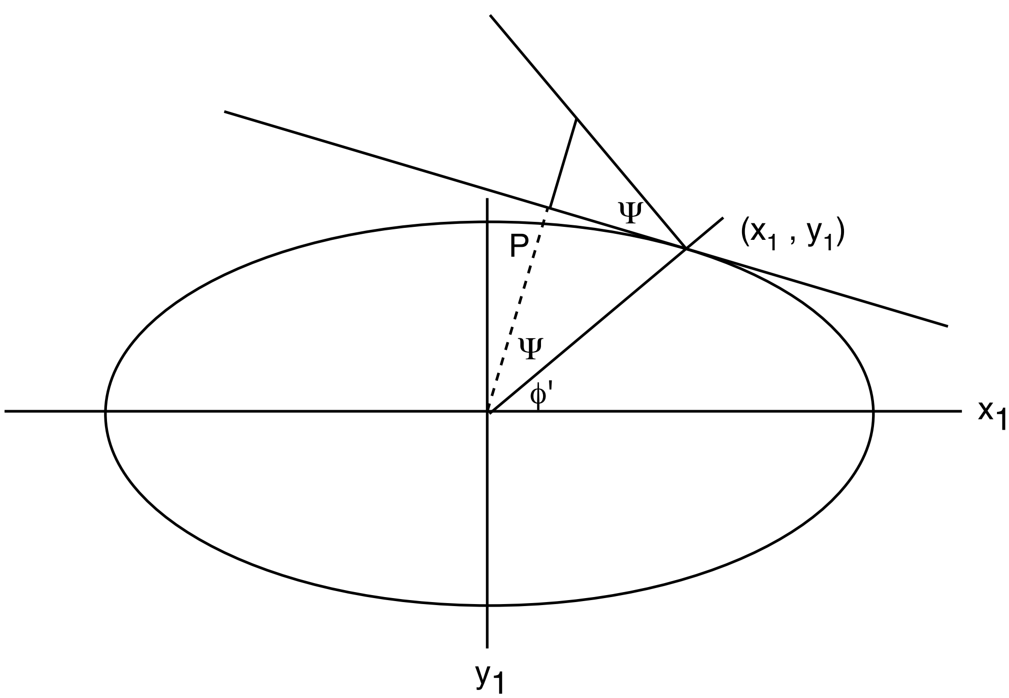
So the final ellipse will have a major axis of 7 and a minor axis of 2, and will be rotated by 25.565°.

This ellipse is drawn on the original diagram, in its rotated and unrotated orientations.

We have seen that for any arbitrary homogeneous (i.e. linear) deformation of a unit circle, the result will be an ellipse. The major and minor axes of this ellipse will not, in general, coincide with the coordinate axes, however the coordinate axes may be rotated to make them coincide. Hence any linear (homogeneous) deformation may be thought of as an irrotational strain, where the directions of greatest and least extension are parallel to the coordinate axes, followed by a rigid body rotation. Because it is usually not possible to detect or measure rigid body rotations in geologically relevant problems, we will now concentrate on irrotational strain.

Consider the deformation of the unit circle shown below:





How does the length of the line, originally oriented at an angle ϕ from the x axis, change as the unit circle is deformed? Originally that line extends from the origin to point (x,y) where $x = \cos\phi$ and $y=\sin\phi$, so its length equals $\sqrt{(\cos^2\phi + \sin^2\phi)} = 1$. (This is good, because that is what the radius of a unit circle should be...) After deformation it extends from the origin to the point (x_1,y_1), where

$$x_1 = x\sqrt{\lambda_1} = \cos(\phi)\sqrt{\lambda_1}$$

$$y_1 = y\sqrt{\lambda_2} = \sin(\phi)\sqrt{\lambda_2}$$

so the length equals $\sqrt{(\lambda_1\cos^2\phi + \lambda_2\sin^2\phi)}$ which we will now call $\sqrt{\lambda}$. It would be more useful to know how much change there has been in the length of a line oriented with respect to the major axis of the strained ellipse (i.e. in terms of ϕ' rather than ϕ). We see that

$$\cos\phi' = \frac{x_1}{\sqrt{\lambda}} \text{ so } x_1 = \cos\phi'\sqrt{\lambda} \text{ and } \sin\phi' = \frac{y_1}{\sqrt{\lambda}} \text{ so } y_1 = \sin\phi'\sqrt{\lambda}$$

And going back to the equations just above, we see that we can substitute to obtain:

$$\cos\phi = \frac{x_1}{\sqrt{\lambda_1}} = \frac{\sqrt{\lambda}\cos\phi'}{\sqrt{\lambda_1}}$$

$$\sin\phi = \frac{y_1}{\sqrt{\lambda_2}} = \frac{\sqrt{\lambda}\sin\phi'}{\sqrt{\lambda_2}}$$

Recalling that $\sin^2\phi + \cos^2\phi = 1$, we see that

$$\frac{\lambda\cos^2\phi'}{\lambda_1} + \frac{\lambda\sin^2\phi'}{\lambda_2} = 1$$

or

$$\frac{1}{\lambda_1}\cos^2\phi' + \frac{1}{\lambda_2}\sin^2\phi' = \frac{1}{\lambda}$$

and letting $\frac{1}{\lambda_1} = \lambda'_1; \quad \frac{1}{\lambda_2} = \lambda'_2; \quad \frac{1}{\lambda} = \lambda'$

$$\lambda' = \lambda'_1\cos^2\phi' + \lambda'_2\sin^2\phi'$$

What is the shear strain associated with the deformation of the line originally at an angle of ϕ with the x axis? Initially the tangent to the circle at point (x,y) made a right angle with this line. After deformation, the tangent to the ellipse at point (x_1',y_1') is no longer perpendicular to the line from the origin to point (x_1',y_1') but is different by an angle, ψ, the angle of shear. We wish to find ψ as a function of ϕ'. We begin by getting the equation for the deformed ellipse into a form where y_1 is a function of x_1 so we will be able to differentiate it to find its slope:

$$\frac{x_1^2}{\lambda_1} + \frac{y_1^2}{\lambda_2} = 1 \quad \text{or} \quad y_1^2 = \lambda_2\left(1 - \frac{x_1^2}{\lambda_1}\right) \quad \text{or} \quad y_1 = \sqrt{\lambda_2 - \frac{\lambda_2 x_1^2}{\lambda_1}}$$

Now the slope of the tangent to the ellipse is given by the derivative:

$$\frac{dy_1}{dx_1} = \frac{1}{2}\left(\lambda_2 - \frac{\lambda_2 x_1^2}{\lambda_1}\right)^{-\frac{1}{2}}\left(-2\frac{\lambda_2}{\lambda_1}x_1\right) = \frac{1}{2}\left(\frac{1}{y_1}\right)\left(-2\frac{\lambda_2}{\lambda_1}x_1\right) \quad \text{or} \quad \frac{dy_1}{dx_1} = -\frac{x_1\lambda_2}{y_1\lambda_1}$$

The tangent line will be a straight line with this slope passing through (x_1',y_1'). As it passes that point, we know its equation will be : $y_1 = -\frac{x_1'\lambda_2}{y_1'\lambda_1}x_1 + k$. Multiply both sides by $\frac{y_1'}{\lambda_2}$ to get

$$\frac{y_1' y_1}{\lambda_2} = -\frac{x_1' x_1}{\lambda_1} + k\frac{y_1'}{\lambda_2} \quad \text{or} \quad \frac{y_1' y_1}{\lambda_2} + \frac{x_1' x_1}{\lambda_1} = k\frac{y_1'}{\lambda_2}$$

Since (x_1',y_1') must be on the ellipse, this equation requires that $k\frac{y_1'}{\lambda_2} = 1$ or $k = \frac{\lambda_2}{y_1'}$.

Recall that $x_1' = (\cos\phi)\sqrt{\lambda_1}$ and that $y_1' = (\sin\phi)\sqrt{\lambda_2}$, and this last equation becomes

$$\frac{x_1 \cos\phi}{\sqrt{\lambda_1}} + \frac{y_1 \sin\phi}{\sqrt{\lambda_2}} = 1$$ as the equation of the tangent line.

Now, let's say we know the equation of a line is y = -mx + b. Then the line perpendicular to this will have a slope of $\frac{1}{m}$ and if it goes through the origin it will have the equation $y = \frac{x}{m}$. These two lines will intersect where $\frac{x}{m} = -mx + b$ or $b = \left(m + \frac{1}{m}\right)x = \frac{m^2+1}{m}x.$

So $x = \frac{bm}{m^2+1}$ hence $y = \frac{x}{m} = \frac{b}{m^2+1}$.

The distance, P, from the origin to this point is

$$P = \sqrt{x^2 + y^2} = \sqrt{\frac{b^2 m^2}{\left(m^2+1\right)^2} + \frac{b^2}{\left(m^2+1\right)^2}} = b\sqrt{\frac{m^2+1}{\left(m^2+1\right)^2}} = \frac{b}{\sqrt{m^2+1}}$$

Substituting our values for b and m:

$$P = \frac{\dfrac{\lambda_2}{y'_1}}{\sqrt{\dfrac{\left(x'_1\lambda_2\right)^2 + \left(y'_1\lambda_1\right)^2}{\left(y'_1\lambda_1\right)^2}}} = \frac{\dfrac{\lambda_2}{y'_1}}{\dfrac{\sqrt{\left(x'_1\lambda_2\right)^2 + \left(y'_1\lambda_1\right)^2}}{y'_1\lambda_1}} = \frac{\lambda_2\lambda_1}{\sqrt{\left(x'_1\lambda_2\right)^2 + \left(y'_1\lambda_1\right)^2}}$$

Since we know that $x_1' = (\cos\phi)\sqrt{\lambda_1}$ and that $y_1' = (\sin\phi)\sqrt{\lambda_2}$,

$$P = \frac{\lambda_2\lambda_1}{\sqrt{\left(\cos^2\phi\right)\lambda_1\lambda_2^2 + \left(\sin^2\phi\right)\lambda_2\lambda_1^2}} = \frac{1}{\sqrt{\dfrac{\left(\cos^2\phi\right)\lambda_1\lambda_2^2 + \left(\sin^2\phi\right)\lambda_2\lambda_1^2}{\left(\lambda_2\lambda_1\right)^2}}} = \frac{1}{\sqrt{\dfrac{\cos^2\phi}{\lambda_1} + \dfrac{\sin^2\phi}{\lambda_2}}}$$

Now, to evaluate Ψ, we recall that $\sin^2\theta + \cos^2\theta = 1$:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2}{\cos^2} = \frac{1}{\cos^2\theta} \quad \text{or} \quad \tan^2\theta + 1 = \frac{1}{\cos^2\theta}$$

since $\cos\Psi = \dfrac{P}{\sqrt{\lambda}}$, $\dfrac{1}{\cos^2\Psi} = \dfrac{\lambda}{P^2}$ and $\tan^2\Psi = \dfrac{\lambda}{P^2} - 1$

Consequently, $\tan^2\Psi = \lambda\left(\dfrac{\cos^2\phi}{\lambda_1} + \dfrac{\sin^2\phi}{\lambda_2}\right) - 1$

Substituting for λ, $\tan^2\Psi = \left(\lambda_1\cos^2\phi + \lambda_2\sin^2\phi\right)\left(\dfrac{\cos^2\phi}{\lambda_1} + \dfrac{\sin^2\phi}{\lambda_2}\right) - 1$

$$\tan^2\Psi = \frac{\lambda_1}{\lambda_1}\cos^4\phi + \frac{\lambda_2}{\lambda_1}\sin^2\cos^2\phi + \frac{\lambda_1}{\lambda_2}\sin^2\cos^2\phi + \frac{\lambda_2}{\lambda_2}\sin^4\phi - 1$$

$$\tan^2\Psi = \cos^4\phi + \frac{\lambda_2}{\lambda_1}\sin^2\cos^2\phi + \frac{\lambda_1}{\lambda_2}\sin^2\cos^2\phi + \sin^4\phi - 1$$

but $1 = \left(\sin^2\phi + \cos^2\phi\right)^2 = \sin^4\phi + 2\sin^2\phi\cos^2\phi + \cos^4\phi$

$$\tan^2\Psi = \frac{\lambda_2}{\lambda_1}\sin^2\cos^2\phi + \frac{\lambda_1}{\lambda_2}\sin^2\cos^2\phi - 2\sin^2\phi\cos^2\phi$$

so $\tan^2\Psi = \left(\dfrac{\lambda_2}{\lambda_1} + \dfrac{\lambda_1}{\lambda_2} - 2\right)\sin^2\phi\cos^2\phi = \dfrac{\lambda_2^2 + \lambda_1^2 - 2\lambda_1\lambda_2}{\lambda_1\lambda_2}\sin^2\phi\cos^2\phi = \dfrac{\left(\lambda_2 - \lambda_1\right)^2}{\lambda_1\lambda_2}\sin^2\phi\cos^2\phi$

Taking the square roots of both sides leaves $\tan\Psi = \dfrac{\lambda_2 - \lambda_1}{\sqrt{\lambda_1\lambda_2}}\sin\phi\cos\phi$.

Once again, we'd prefer that our results were in terms of ϕ' instead of ϕ, so we'll again substitute:

$$\cos\phi = \frac{x_1}{\sqrt{\lambda_1}} = \frac{\sqrt{\lambda}\cos\phi'}{\sqrt{\lambda_1}}$$

$$\sin\phi = \frac{y_1}{\sqrt{\lambda_2}} = \frac{\sqrt{\lambda}\sin\phi'}{\sqrt{\lambda_2}}$$

$$\tan\Psi = \frac{\lambda_2 - \lambda_1}{\sqrt{\lambda_1\lambda_2}}\frac{\sqrt{\lambda}\cos\phi'}{\sqrt{\lambda_1}}\frac{\sqrt{\lambda}\sin\phi'}{\sqrt{\lambda_2}} = \lambda\frac{(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2}\sin\phi'\cos\phi'$$

Recall that by definition, the shear strain, γ, is equal to tan Ψ. Then, for what will eventually be seen as a convenience, we will define an otherwise obscure parameter, γ' as being equal to $\dfrac{\gamma}{\lambda}$. Combining this with our previously defined $\lambda' = \dfrac{1}{\lambda}$, etc., we arrive at:

$$\gamma' = (\lambda_1' - \lambda_2')\sin\phi'\cos\phi'$$

earlier we found that $\lambda' = \lambda_1'\cos^2\phi' + \lambda_2'\sin^2\phi$

Looks like a place to employ some double angle formulas. We remember that sin (a + b) = sin a cos b + sin b cos a, and that cos (a + b) = cos a cos b - sin a sin b, which leads to

$$\sin 2\phi' = 2\sin\phi'\cos\phi'$$

$$\cos 2\phi' = \cos^2\phi' - \sin^2\phi' = 1 - 2\sin^2\phi' = 2\cos^2\phi' - 1$$

$$\sin^2\phi' = \frac{1 - \cos 2\phi'}{2} \text{ and } \cos^2\phi' = \frac{1 + \cos 2\phi'}{2}$$

and we can substitute these into earlier results to obtain

$$\gamma' = \frac{(\lambda_1' - \lambda_2')}{2}\sin 2\phi'$$

$$\lambda' = \lambda_1'\frac{1 + \cos 2\phi'}{2} + \lambda_2'\frac{1 - \cos 2\phi'}{2} = \frac{(\lambda_1' + \lambda_2')}{2} + \frac{(\lambda_1' - \lambda_2')}{2}\cos 2\phi'$$

These are parametric equations for a circle in λ', γ' space, having a radius of $\dfrac{\lambda_1' - \lambda_2'}{2}$ and centered on the λ' axis at $\lambda' = \dfrac{\lambda_1' + \lambda_2'}{2}$